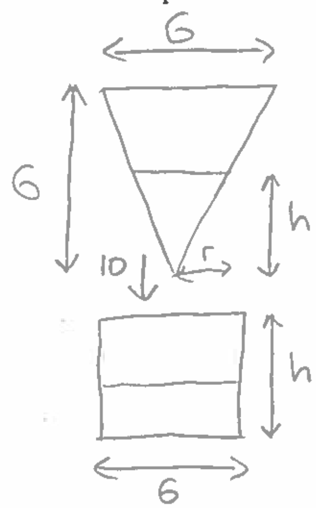
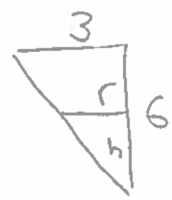
Of questions 10 – 13, you must answer **three questions**. If you answer more than three then only **your best three scores will be counted**.

10. (12 marks) Coffee is draining from a conical filter, of height and diameter 6cm, into a cylindrical coffee pot, also of diameter 6cm, at a rate of $10\text{cm}^3/\text{min}$. Calculate how fast the levels in both the pot and the cone are changing when the coffee in the cone is 5cm deep.



Pot

$$V = \pi r^2 h$$
$$= 9\pi h$$
$$\frac{dV}{dt} = 9\pi \frac{dh}{dt}$$
$$\Longrightarrow \frac{dh}{dt} = \frac{1}{9\pi}\frac{dV}{dt} = \boxed{\frac{10}{9\pi}\text{ cm}^3/\text{min}}$$

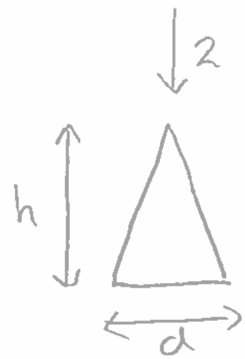


$$\frac{r}{h} = \frac{3}{6}$$
$$\Longrightarrow r = \frac{h}{2}$$

Cone

$$V = \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h$$
$$= \frac{\pi}{12}h^3$$
$$\frac{dV}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt}$$
$$\Longrightarrow \frac{dh}{dt} = \frac{4}{h^2\pi}\frac{dV}{dt} = \frac{-40}{h^2\pi}$$
$$h = 5 \Longrightarrow \frac{dh}{dt} = \frac{-40}{25\pi} = \boxed{\frac{-8}{5\pi}\text{ cm}^3/\text{min}}$$

11. (12 marks) Assume that sand allowed to pour onto a level surface will form a pile in the shape of a cone, with height equal to the diameter of its base. If sand is poured at $2\text{m}^3/\text{sec}$, how fast is the height of the pile changing when its base is 8m in diameter.



$$d = h \implies r = \frac{h}{2}.$$

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h$$

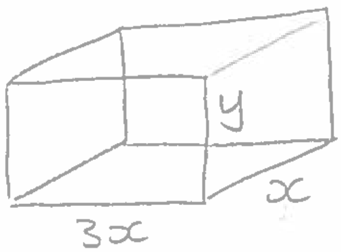
$$= \frac{\pi}{12}h^3$$

$$\implies \frac{dV}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt}$$

$$\implies \frac{dh}{dt} = \frac{4}{h^2\pi}\frac{dV}{dt} = \frac{8}{h^2\pi}$$

$$d = 8 \implies h = 8 \implies \frac{dh}{dt} = \frac{8}{8^2\pi} = \boxed{\frac{1}{8\pi}\,m^3/sec}$$

12. (12 marks) We want to construct a box whose base length is 3 times the base width. The material used to build the top and bottom cost \$10 per square foot and the material used to build the sides cost \$6 per square foot. If the box must have a volume of 50ft^3, determine the dimensions that will minimise the cost to build the box.



$$C = 6(2xy + 2(3xy)) + 10(2)(3x^2)$$
$$= 48xy + 60x^2$$

$$V = 3x \times x \times y = 3x^2y = 50$$
$$\Longrightarrow y = \frac{50}{3x^2}$$

$$\Longrightarrow C = 48x\left(\frac{50}{3x^2}\right) + 60x^2$$
$$= \frac{16\times 50}{x} + 60x^2$$

$$\Longrightarrow C' = -\frac{16\times 50}{x^2} + 120x = 0$$

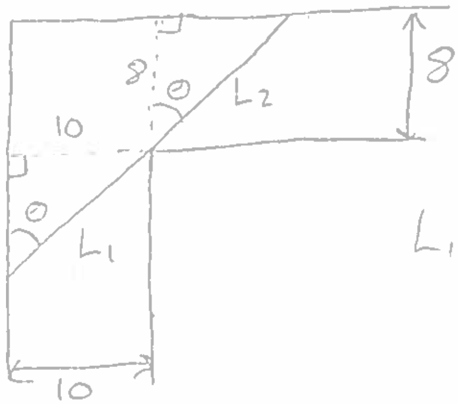
$$\Longrightarrow 120x = \frac{16\times 50}{x^2}$$

$$\Longrightarrow x^3 = \frac{16\times 50}{120} = \frac{16\times 5}{12}$$
$$= \frac{4\times 5}{3} = \frac{20}{3}$$

$$\Longrightarrow \boxed{x = \sqrt[3]{\frac{20}{3}}}$$

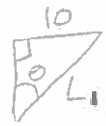
$$\Longrightarrow \boxed{y = \frac{50}{3\sqrt[3]{\frac{20}{3}}^{2}}} \left(= \frac{50}{\sqrt[3]{1200}} \right.$$
$$= \sqrt[3]{\frac{125000}{1200}}$$
$$\left. = 5\sqrt[3]{\frac{5}{6}} \right)$$

13. (12 marks) A piece of pipe is being carried down a hallway that is 10 feet wide. At the end of the hallway there is a right-angled turn and the hallway narrows to 8 feet wide. What is the longest pipe that can be carried (always keeping it horizontal) around the turn in the hallway?



$L = L_1 + L_2$

L_1:



$\sin(\theta) = \frac{\text{opp}}{\text{adj}} = \frac{10}{L_1}$

$\implies L_1 = \frac{10}{\sin(\theta)} = 10\csc(\theta)$

L_2:

$\cos(\theta) = \frac{\text{adj}}{\text{hyp}} = \frac{8}{L_2}$

$\implies L_2 = \frac{8}{\cos(\theta)} = 8\sec(\theta)$

$L = 10\csc(\theta) + 8\sec(\theta)$

$L' = -10\csc(\theta)\cot(\theta) + 8\sec(\theta)\tan(\theta) = 0$

$\implies 10\csc(\theta)\cot(\theta) = 8\sec(\theta)\tan(\theta)$

$\implies 10\csc(\theta) = 8\sec(\theta)\tan^2(\theta)$

$\implies 10 = 8\tan^3(\theta)$

$\implies \tan^3(\theta) = \frac{5}{4} \implies \boxed{\theta = \tan^{-1}\left(\sqrt[3]{5/4}\right)}$

$\implies L = 10\csc\left(\tan^{-1}\left(\sqrt[3]{5/4}\right)\right) + 8\sec\left(\tan^{-1}\left(\sqrt[3]{5/4}\right)\right)$